A note on free choice and exclusion-based presuppositional exhaustification

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Abstract

In Del Pinal, Bassi & Sauerland (2024) we present a novel Grammatical theory of FC based on a presuppositional exhaustification operation, called '**pex**', and show that it resolves various puzzles concerning the way in which FC and related sentences project from various embedded environments. In that paper, we derive basic FC effects by assuming that **pex** is sensitive to both innocently excludable (IE) and includable (II) alternatives (call this '$\mathbf{pex}^{IE+II}$'). This note is intended as an addendum to our main paper: we show that we can derive the same basic results using a purely IE based presupositional exhaustification operator (call this $\mathbf{pex}^{IE}$). Relative to the target readings, $\mathbf{pex}^{IE+II}$ and $\mathbf{pex}^{IE}$ are descriptively on equal footing. Yet there are non-trivial theoretical trade-offs: while the derivation of basic FC effects is simpler with $\mathbf{pex}^{IE+II}$—in that it doesn't require recursive exhaustification—$\mathbf{pex}^{IE}$ uses simpler operations which are sensitive only to IE alternatives. In this note, we present two models to derive FC effects with $\mathbf{pex}^{IE}$: one appeals to a local accommodation operator and the other uses a Strong Kleene semantics.

Keywords: free choice, scalar implicatures, exhaustification, presuppositions, presupposed free choice, filtering free choice, accommodation, pragmatics.

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1 Free choice and related effects with IE-based pex

In Del Pinal, Bassi & Sauerland (2024), we used $\mathbf{pex}^{IE+II}$ to derive the FC reading of $\Diamond\vee$-sentences, the double prohibition reading of $\neg\Diamond\vee$-sentences, the double requirement reading of $\Box\wedge$-sentences and the negative FC reading of $\neg\Box\wedge$-sentences. The key point is not just that we get the standard entailments for those sentence—other theories also get that—but that we get those entailments via specific combinations of assertive vs. non-at issue/presupposed components:

(1) a. $\mathbf{pex}^{IE+II}[\Diamond(p\vee q)] = \Diamond(p\vee q)_{\Diamond p\leftrightarrow\Diamond q}$ $\models \Diamond p\wedge\Diamond q$
b. $\neg\mathbf{pex}^{IE+II}[\Diamond(p\vee q)] = \neg\Diamond(p\vee q)_{\Diamond p\leftrightarrow\Diamond q}$ $\models \neg\Diamond p\wedge\neg\Diamond q$
c. $\mathbf{pex}^{IE+II}[\Box(p\wedge q)] = \Box(p\wedge q)_{\Box p\leftrightarrow\Box q}$ $\models \Box p\wedge\Box q$
d. $\neg\mathbf{pex}^{IE+II}[\Box(p\wedge q)] = \neg\Box(p\wedge q)_{\Box p\leftrightarrow\Box q}$ $\models \neg\Box p\wedge\neg\Box q$

Our main goal in Del Pinal et al. (2024) was to show that the structuring into presupposed vs assertive components shown in (1) solves various puzzles concerning how these sentences behave in certain embedded environments. Our aim here is to show that those results can also be derived with just $\mathbf{pex}^{IE}$. However, if we use $\mathbf{pex}^{IE}$, we need to use in each case recursive exhaustification. One way of getting the target results is to use an intermediate application of the local accommodation operator, ACC, as captured in (2) (where $\text{ACC}[p_q] = q\wedge p$):

(2) a. $\mathbf{pex}^{IE+II}[\Diamond(p\vee q)]$ $\Leftrightarrow \mathbf{pex}^{IE}[\text{ACC}[\mathbf{pex}^{IE}[\Diamond(p\vee q)]]]$
b. $\neg\mathbf{pex}^{IE+II}[\Diamond(p\vee q)]$ $\Leftrightarrow \neg\mathbf{pex}^{IE}[\text{ACC}[\mathbf{pex}^{IE}[\Diamond(p\vee q)]]]$
c. $\mathbf{pex}^{IE+II}[\Box(p\wedge q)]$ $\Leftrightarrow \mathbf{pex}^{IE}[\text{ACC}[\mathbf{pex}^{IE}[\Box(p\wedge q)]]]$
d. $\neg\mathbf{pex}^{IE+II}[\Box(p\wedge q)]$ $\Leftrightarrow \neg\mathbf{pex}^{IE}[\text{ACC}[\mathbf{pex}^{IE}[\Box(p\wedge q)]]]$

The parses on the right-hand side, with recursive application of $\mathbf{pex}^{IE}$ and an intermediate ACC, are more complex than the ones on the left, with a single application of $\mathbf{pex}^{IE+II}$. Still, some semanticists may prefer analyses based on $\mathbf{pex}^{IE}$, for it is based on more standard assumptions about the operations of covert exhaustification, namely, that it is only sensitive to IE alternatives. In this first section, we derive the results in (2), and show that the intermediate insertion of ACC is warranted on standard assumptions, namely, that the corresponding parses without ACC result in LFs which are defective on independent grounds.

1.1 FC-$\Diamond\vee$ and DP-$\neg\Diamond\vee$-sentences

Let us begin by deriving the FC reading of $\Diamond\vee$-sentences. To appreciate why we need an intermediate ACC, consider first the parse in (3a), which is analogous to the one used by standard $\mathbf{exh}^{IE}$-based theories. The problem when we compute $\mathbf{pex}_1^{IE}$ is that—unlike accounts with $\mathbf{exh}^{IE}$— the disjunctive alternatives of its prejacent, i.e., $\mathbf{pex}_2^{IE}\ [\Diamond p]$ and $\mathbf{pex}_2^{IE}\ [\Diamond q]$ are not in IE. For due to the projection of presuppositions under negation, captured in (3c)-(3d), these alternatives can't jointly be part of any maximal subset of (negated) alternatives that is consistent with $\mathbf{pex}_2^{IE}\ [\Diamond[p\vee q]]$, as captured in (3f).[1] This result holds independently of whether $\mathbf{pex}_2^{IE}\ [\Diamond[p\vee q]] = \Diamond[p\vee q]$, as we assume for simplicity in (3e), or relevance at a context is such that $\mathbf{pex}_2^{IE}\ [\Diamond[p\vee q]] = \Diamond[p\vee q]_{\neg\Diamond[p\wedge q]}$.

(3) a. $\mathbf{pex}_1^{IE}\ [\mathbf{pex}_2^{IE}\ [\Diamond[p\vee q]]]$

b. $\text{Alt}(\mathbf{pex}_2^{IE}\ [\Diamond[p\vee q]]) = \left\{ \begin{array}{l} \mathbf{pex}_2^{IE}[\Diamond[p\vee q]], \\ \mathbf{pex}_2^{IE}[\Diamond p], \mathbf{pex}_2^{IE}\ [\Diamond q], \\ \mathbf{pex}_2^{IE}[\Diamond[p\wedge q]] \end{array} \right\}$

c. $\neg\ \mathbf{pex}_2^{IE}\ [\Diamond p] = \neg[\Diamond p_{\neg\Diamond q}] = \neg\Diamond p_{\neg\Diamond q}$

d. $\neg\ \mathbf{pex}_2^{IE}\ [\Diamond q] = \neg[\Diamond q_{\neg\Diamond p}] = \neg\Diamond q_{\neg\Diamond p}$

e. $\mathbf{pex}_2^{IE}\ [\Diamond[p\vee q]] = \Diamond[p\vee q]$

f. $\Diamond[p\vee q]\wedge\neg\Diamond p_{\neg\Diamond q}\wedge\neg\Diamond q_{\neg\Diamond p} = \bot$

Now, although we don't get the FC reading on this parse, $\mathbf{pex}_1^{IE}$ isn't entirely vacuous, since it could still negate the $\mathbf{pex}_2^{IE}\ [\Diamond[p\wedge q]]$ alternative. Yet this readings for $\Diamond\vee$-sentences—namely, $\Diamond[p\vee q]_{\neg\Diamond[p\wedge q]}$—could also be obtained via the structurally simpler parse $\mathbf{pex}^{IE}\ [\Diamond[p\vee q]]$. So the parse in (3a) would be ruled out by any reasonable principle of structural economy.

The FC readings of $\Diamond\vee$-sentences can be derived with $\mathbf{pex}^{IE}$ by adopting a slightly different parse with intermediate ACC between the two applications of $\mathbf{pex}^{IE}$, as in (4a). This parse is licensed because ACC is introduced to rescue a parse that would otherwise be defective, as we just established for (3a). The parse in (4a) generates the FC reading of $\Diamond\vee$-sentences. Due to the effect of the embedded ACC, negation of the disjunctive alternatives for the prejacent of $\mathbf{pex}_1^{IE}$ now has the effect of generating the homogeneity proposition $\Diamond p \leftrightarrow \Diamond q$, as shown in (4d). Adding that homogeneity proposition as a presupposition to the prejacent, $\Diamond[p\vee q]$, guarantees the FC entailments, as captured in (4f).

(4) a. $\mathbf{pex}_1^{IE}\ [\text{ACC}[\mathbf{pex}_2^{IE}\ [\Diamond[p\vee q]]]$

1 The result in (3f) guarantees that no other reasonable notion of exclusion would generate the FC inferences via negation of $\mathbf{pex}_2\ [\Diamond p]$ and $\mathbf{pex}_2\ [\Diamond p]$.

b. $\text{Alt}(\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond[p \vee q]]]) = \begin{cases} \textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond[p \vee q]]], \\ \textsc{acc}[\mathbf{pex}_2\ [\Diamond p]], \textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond q]], \\ \textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond[p \wedge q]]] \end{cases}$

c. $[\![(4a)]\!] = \begin{cases} \textbf{ps: } \neg\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond p]] \wedge \neg\textsc{acc}[\mathbf{pex}_2\ [\Diamond q]] \\ \textbf{asserts: } \textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond[p \vee q]]] \end{cases}$

d. $\neg\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond p]] \wedge \neg\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond q]]$
$= \neg[\neg\Diamond q \wedge \Diamond p] \wedge \neg[\neg\Diamond p \wedge \Diamond q]$
$= [\Diamond q \vee \neg\Diamond p] \wedge [\Diamond p \vee \neg\Diamond q] = [\Diamond p \rightarrow \Diamond q] \wedge [\Diamond q \rightarrow \Diamond p]$
$= \Diamond p \leftrightarrow \Diamond q$

e. $\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond[p \vee q]]] = \Diamond[p \vee q]$

f. $[\![(4a)]\!] = \Diamond[p \vee q]_{\Diamond p \leftrightarrow \Diamond q}$ $\models \Diamond p \wedge \Diamond q$

We should mention two things about the presupposed and assertive content of (4a), as captured in (4c). First, we get the good result that, in the **asserts** part of (4c), $\mathbf{pex}_2^{IE}$ can't innocently exclude the $\Diamond p$ nor the $\Diamond q$ alternatives to its prejacent $\Diamond[p \vee q]$: for since $\neg\Diamond p$ and $\neg\Diamond q$ can't both be consistently conjoined with $\Diamond[p \vee q]$, they can't jointly be part of any maximal subset of (negated) alternatives that is consistent with $\Diamond[p \vee q]$. Secondly, when calculating the **ps** content triggered by $\mathbf{pex}_1^{IE}$, we ignored the conjunctive alternative $\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond[p \wedge q]]]$. Yet if we are interested in cases in which that alternative is relevant, we get the result that it is excludable, and ultimately resolves to adding $\neg\Diamond[p \wedge q]$ to the **ps** part of (4c).[2] In short, we have derived the following results:

(5) If $\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond[p \wedge q]]] \notin Alt \cap R$ for $\mathbf{pex}_1$:

a. $[\![\mathbf{pex}_1^{IE}\ [\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond[p \vee q]]]]]\!] = \begin{cases} \textbf{ps: } \Diamond p \leftrightarrow \Diamond q \\ \textbf{asserts: } \Diamond[p \vee q] \end{cases}$

(6) If $\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond[p \wedge q]]] \in Alt \cap R$ for $\mathbf{pex}_1$:

a. $[\![\mathbf{pex}_1^{IE}\ [\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond[p \vee q]]]]]\!] = \begin{cases} \textbf{ps: } \Diamond p \leftrightarrow \Diamond q \wedge \neg\Diamond[p \wedge q] \\ \textbf{asserts: } \Diamond[p \vee q] \end{cases}$

2 To be clear, $\neg\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond[p \wedge q]]]$ resolves to $\neg\Diamond[p \wedge q]$ assuming here that the target focus for $\mathbf{pex}_2^{IE}$ doesn't include the modal operator. To predict the correct behavior for embedded $\Diamond\vee$-sentences in the presupposed & filtering FC puzzles, deriving $\neg\Diamond[p \wedge q]$ as part of the presuppositional content, as we have done here, is desirable. But couldn't $\neg\Diamond[p \wedge q]$ be added when we compute the output of $\mathbf{pex}_2^{IE}$ in the **asserts** part of (4c), which due to its being embedded under an ACC operator, would mean that $\neg\Diamond[p \wedge q]$ ends up as part of the assertive content? Are we just stipulating in which structural position the $\neg\Diamond[p \wedge q]$ is introduced? No, for our assumption follows from independent principles: $\neg\Diamond[p \wedge q]$ isn't generated twice when computing recursive exhaustification because that would be redundant and inefficient, and it is generated when computing $\mathbf{pex}_1^{IE}\ [\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond[p \vee q]]]$ rather than $\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Diamond[p \vee q]]$ because that option triggers stronger presuppositions (so is preferred given something like a 'Maximize Presuppositions' principle).

This captures the observation, due to Simons (2005), that $\Diamond\vee$-sentences have two subtly different FC readings, one in which is compatible with $\Diamond p$, $\Diamond q$ and $\Diamond(p \wedge q)$, and one in which is compatible with $\Diamond p$, $\Diamond q$ but not with $\Diamond(p \wedge q)$. The first reading is captured by (5a) and the second by (6a). Since the difference between (5a) and (6a) is irrelevant for the puzzles considered in Del Pinal, Bassi & Sauerland (2024), here we focus for simplicity on the FC reading in (5a).

Consider next the double prohibition (DP) reading of $\neg\Diamond\vee$-sentences. In **exh**-based systems, recall, the DP reading for (7) follows from economy. When a parse with **exh** leads to an overall weakening of meaning compared to a parallel parse without **exh**, the latter is preferred. For $\neg\Diamond\vee$-sentences, the resulting parse without **exh** under negation supports the DP reading. In a $\textbf{pex}^{IE}$-based theory, in contrast, we can derive the DP reading given the parse in (7). For due to the presuppositional structure of FC, the negated part only directly affects the assertive output of $\textbf{pex}^{IE}$, which gets us precisely the DP inferences.

(7) $\neg[\textbf{pex}^{IE}\ [\Diamond[p \vee q]]$
$= \neg[\Diamond[p \vee q]_{\Diamond p \leftrightarrow \Diamond q}]$
$= \neg\Diamond[p \vee q]_{\Diamond p \leftrightarrow \Diamond q}$ $\qquad \models \neg\Diamond \mathrm{p} \wedge \neg\Diamond \mathrm{q}$

1.2 $\Box\wedge$ and negative FC-$\neg\Box\wedge$-sentences

Moving next to $\Box\wedge$-sentences, the situation is roughly analogous to $\Diamond\vee$-sentences, in the sense that recursive application of $\textbf{pex}^{IE}$ is vacuous, hence banned by economy, whereas a subtle yet non-trivial effect obtains if we insert an intermediate ACC. The prejacent of $\textbf{pex}^{IE}_1$ has two *IE* alternatives, $\text{ACC}[\textbf{pex}^{IE}_2\ [\Box p]]$ and $\text{ACC}[\textbf{pex}^{IE}_2\ [\Box q]]$, hence the negation of each has to be added at the presuppositional level, as in (8c). As shown in (8d), the conjunction of those negated alternatives, $\neg\text{ACC}[\textbf{pex}^{IE}_2\ [\Box p]] \wedge \neg\text{ACC}[\textbf{pex}^{IE}_2\ [\Box q]]$, is equivalent to the homogeneity proposition $\Box p \leftrightarrow \Box q$.

(8) a. $\textbf{pex}^{IE}_1\ [\text{ACC}[\textbf{pex}^{IE}_2\ [\Box[p \wedge q]]]$

b. $\text{Alt}(\text{ACC}[\textbf{pex}^{IE}_2\ [\Box[p \wedge q]]])= \left\{ \begin{array}{l} \text{ACC}[\textbf{pex}^{IE}_2\ [\Box[p \wedge q]]], \\ \text{ACC}[\textbf{pex}^{IE}_2\ [\Box p]],\ \text{ACC}[\textbf{pex}^{IE}_2\ [\Box q]], \\ \text{ACC}[\textbf{pex}^{IE}_2\ [\Box[p \vee q]]] \end{array} \right.$

c. $[\![(8)]\!] = \left\{ \begin{array}{l} \textbf{ps: } \neg\text{ACC}[\textbf{pex}^{IE}_2\ [\Box p]] \wedge \neg\text{ACC}[\textbf{pex}^{IE}_2\ [\Box q]] \\ \textbf{asserts: } \text{ACC}[\textbf{pex}^{IE}_2\ [\Box[p \wedge q]]] \end{array} \right.$

d. $\neg\text{ACC}[\textbf{pex}^{IE}_2\ [\Box p]] \wedge \neg\text{ACC}[\textbf{pex}^{IE}_2\ [\Box q]]$
$= \neg[\Box p \wedge \neg\Box q] \wedge \neg[\Box q \wedge \neg\Box p]$
$= [\neg\Box p \vee \Box q] \wedge [\neg\Box q \vee \Box p]$
$= [\Box p \rightarrow \Box q] \wedge [\Box q \rightarrow \Box p]$
$= \Box p \leftrightarrow \Box q$

e. $\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Box[p \wedge q]]] = \Box[p \wedge q]$
f. $[\![(8)]\!] = \Box[p \wedge q]_{\Box p \leftrightarrow \Box q}$

The homogeneity presupposition is entailed by the assertive component; hence doesn't strengthen the overall entailments of $\Box\wedge$-sentences in unembedded contexts. However, it helps derive the target readings for various complex sentences that have embedded occurrences of $\Box\wedge$-sentences, as we show in Del Pinal, Bassi & Sauerland (2024). That is connected to the fact that, on this analysis, we can derive a negative FC reading for $\neg\Box\wedge$-sentences by recursively application of $\mathbf{pex}^{IE}$ under negation, as in (9):

(9) $\neg\, \mathbf{pex}_1^{IE}\ [\textsc{acc}[\mathbf{pex}_2^{IE}\ [\Box[p \wedge q]]]$
$= \neg[\Box[p \wedge q]_{\Box p \leftrightarrow \Box q}]$
$= \neg\Box[p \wedge q]_{\Box p \leftrightarrow \Box q}$
$= [\neg\Box p \vee \neg\Box q]_{\Box p \leftrightarrow \Box q}$ $\qquad \models \neg\Box p \wedge \neg\Box q$

A $\mathbf{pex}^{IE}$-based theory also supports a derivation of the negative FC reading of $\neg\Box\wedge$-sentences based on the parse in (10), which is closer to the one used by standard **exh**-based theories. The prejacent of $\mathbf{pex}_1^{IE}$ has two *IE* alternatives, $\textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box p]]$ and $\textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box q]]$ (assuming for simplicity that $\textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box[p \vee q]]]$ is not relevant). Once the conjoined negation of those alternatives is added at the presuppositional level, we get an interpretation for (10) as in (10c). Given the equivalences in (10d) for the **ps** part and in (10e) for the **asserts** part, (10c) reduces to (10f), which entails the target negative FC conjunction, $\neg\Box p \wedge \neg\Box q$.

(10) a. $\mathbf{pex}_1^{IE}\ [\textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box[p \wedge q]]]$

b. $\mathrm{Alt}(\textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box[p \wedge q]]]) = \left\{ \begin{array}{l} \textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box[p \wedge q]]], \\ \textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box p]],\ \textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box q]], \\ \textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box[p \vee q]]] \end{array} \right.$

c. $[\![(10)]\!] = \left\{ \begin{array}{l} \mathbf{ps:}\ \neg\textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box p]] \wedge \neg\textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box q]] \\ \mathbf{asserts:}\ \textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box[p \wedge q]]] \end{array} \right.$

d. $\neg\textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box p]] \wedge \neg\textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box q]]$
$= \neg[\neg\Box p \wedge \Box q] \wedge \neg[\neg\Box q \wedge \Box p]$
$= [\Box p \vee \neg\Box q] \wedge [\Box q \vee \neg\Box p]$
$= [\Box q \rightarrow \Box p] \wedge [\Box p \rightarrow \Box q]$
$= \Box q \leftrightarrow \Box p$

e. $\textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box[p \wedge q]]] = \neg\Box[p \wedge q] = \neg\Box p \vee \neg\Box q$

f. $[\![(10)]\!] = [\neg\Box p \vee \neg\Box q]_{\neg\Box p \leftrightarrow \neg\Box q}$ $\qquad \models \neg\Box p \wedge \neg\Box q$

It is easy to check that, when the $\textsc{acc}[\mathbf{pex}_2^{IE}\ [\neg\Box[p \vee q]]]$ alternative is relevant, we ultimately also add to (10f) the entailment that $\Box[p \vee q]$.

2 Recursive pex and Free Choice in a Strong Kleene Semantics

In this section, we show that $\mathbf{pex}^{IE}$ allows the derivation of FC inferences via recursive exhaustification but using, instead of an (intermediate) local accommodation operator, a Strong Kleene semantics to calculate presupposition projection. First we introduce the notation we use for the exposition, then we introduce Strong Kleene semantics, and how to apply it to recursive $\mathbf{pex}^{IE}$. We then show that positive and negative FC follow from the system.

2.1 The Fraction Notation for presuppositions

Assume that D_{st} is the domain of propositions, i.e. total functions from possible worlds to truth values. We adopt the fraction notation for partial functions Harbour (2014) proposes:

(1) $$\frac{a}{p} = \lambda w \in \{w \in D_s \mid p(w)\} \ . \ a(w) \text{ for } a, p \in D_{st}$$

Also assume $\mathbf{1} = \lambda w \in D_s \ . \ 1$, $\mathbf{0} = \lambda w \in D_s \ . \ 0$ and $\mathbf{\#} = \lambda w \in \emptyset \ . \ 0$. Then these corollaries hold, as Harbour notes:[3]

(2) $$\frac{a}{\mathbf{1}} = a \text{ and } \frac{a}{\mathbf{0}} = \mathbf{\#} \text{ for any } a$$

2.2 Strong Kleene Semantics

A number of recent works argue that the Strong Kleene semantics gives rise to an attractive account of presupposition projection (Fox 2013, George 2008, a.o.). The basic intuition of the Strong Kleene semantics is that a third truth value # represents an indeterminate truth value. If # occurs in the argument of a logical operator, the result is therefore predictable from inspecting the results of replacing # with 0 or 1. If both replacements result in the same truth value b, the indeterminacy of # is irrelevant, and b is the result of the whole. Otherwise though the indeterminacy remains and # is the result. For example, $1\blacktriangle\# = \#$ and $0\blacktriangle\# = 0$ where $\blacktriangle$ represents Strong Kleene conjunction.

The fraction notation can capture trivalent propositions by the following convention:

(3) $$\frac{a}{p}(w) = \begin{cases} \# & \text{if not } p(w) \\ 1 & \text{if } p(w) \text{ and } a(w) \\ 0 & \text{otherwise} \end{cases}$$

3 Note that one of us (Sauerland 2005) used a fraction notation for presuppositions but with opposite roles for numerator and denominator. Harbour's corollaries make his proposal more intuitive, which is why we adopt it.

For trivalent propositions, general Strong Kleene conjunction ▲ and disjunction ▼ amount to the following in the fraction notation:

(4) $$\frac{a}{a'} \blacktriangle \frac{b}{b'} = \frac{a \wedge b}{(a' \wedge b') \vee (a' \wedge \neg a) \vee (b' \wedge \neg b)}$$

(5) $$\frac{a}{a'} \blacktriangledown \frac{b}{b'} = \frac{a \vee b}{(a' \wedge b') \vee (a' \wedge a) \vee (b' \wedge b)}$$

2.3 Recursive Presuppositional Exhaustification and Strong Kleene

Via the recursive application of $\mathbf{pex}^{IE}$, our analysis gives rise to cases of double fractions, where the assertion or the presupposition of a complex term has presuppositions of it own. The first case is easy: When the assertion has a presupposition, we want to just conjoin that presupposition with the presupposition of the more complex term.

(6) $$\frac{\frac{a}{b}}{c} = \frac{a}{b \wedge c} \text{ for } a, b, c \in D_{st}$$

The second case is more difficult: What if the presupposition itself has a presupposition, i.e. $\frac{b}{b'}$ in the denominator. When b' is not satisfied, this intuitively corresponds to uncertainty of whether the result of the complex term is uncertain. We assume that the Strong Kleene interpretation is to use the value of a unless we are sure the presupposition is false. I.e. we assume that only the worlds where $\frac{b}{b'}$ is false are not in the domain of the double fraction. For the other worlds, the truth value of the double fraction is defined as that of a:[4]

(7) $$\frac{a}{\frac{b}{b'}} = \frac{a}{b \vee \neg b'} = \frac{a}{b' \rightarrow b}$$

2.4 Positive Free Choice

Consider now the basic case of FC below, where we assume for simplicity that conjunctive alternatives are not active. In other words, we assume that $C_1 = \{\Diamond(A \vee B), \Diamond A, \Diamond B\}$ and $C_2 = \{\mathbf{pex}^{IE}_{C_1} \Diamond(A \vee B), \mathbf{pex}^{IE}_{C_1} \Diamond A, \mathbf{pex}^{IE}_{C_1} \Diamond B\}$:

(8) $$\mathbf{pex}^{IE}_{C_2}\ \mathbf{pex}^{IE}_{C_1}\ \Diamond(A \vee B)$$

4 Assumption (7) has non-trivial implications for other examples with two presupposition triggers in a single sentence. It is beyond the present scope to evaluate these in detail.

The elements of C_2 are the following propositions (here and in the following we disregard the conjunctive alternative for simplicity):

(9) $$\mathbf{pex}^{IE}_{C_1}\ \Diamond A = \frac{\Diamond A}{\neg\Diamond B}$$

(10) $$\mathbf{pex}^{IE}_{C_1}\ \Diamond B = \frac{\Diamond B}{\neg\Diamond A}$$

The fact that there are two exclusions, (9) and (10), results in another theoretical choice. In a Strong Kleene system, we must combine multiple exclusions with $\blacktriangle$ instead of $\wedge$, but $\blacktriangle$ exhibits more sensitivity to its method of application than $\wedge$. Consider first the result when the exclusions are combined by Strong Kleene conjunction in the presupposition. As the following computation shows, this would maximize the exclusion and result in a contradictory function (specifically, the presupposition and assertion contradict each other, and hence the result is a contradictory proposition with a presupposition):

(11) $$\mathbf{pex}^{IE}_{C_2}\ \mathbf{pex}^{IE}_{C_1}\ \Diamond(A \vee B) = \frac{\Diamond(A \vee B)}{\neg\frac{\Diamond A}{\neg\Diamond B}\blacktriangle\neg\frac{\Diamond B}{\neg\Diamond A}}$$

(12) $$= \frac{\Diamond(A \vee B)}{\frac{\neg\Diamond A}{\neg\Diamond B}\blacktriangle\frac{\neg\Diamond B}{\neg\Diamond A}}$$

(13) $$= \frac{\Diamond(A \vee B)}{\frac{\neg\Diamond A\wedge\neg\Diamond B}{(\neg\Diamond A\wedge\neg\Diamond B)\vee(\neg\Diamond A\vee\Diamond B)\vee(\Diamond A\vee\neg\Diamond B)}}$$

(14) $$= \frac{\Diamond(A \vee B)}{\frac{\neg\Diamond A\wedge\neg\Diamond B}{(\neg\Diamond A\wedge\neg\Diamond B)\vee(\neg\Diamond A\vee\Diamond B)\vee(\Diamond A\vee\neg\Diamond B)}}$$

(15) $$= \frac{\Diamond(A \vee B)}{(\neg\Diamond A\wedge\neg\Diamond B)\vee(\neg\Diamond A\vee\Diamond B)\vee(\Diamond A\vee\neg\Diamond B)}$$

(16) $$= \frac{0}{(\neg\Diamond A\wedge\neg\Diamond B)\vee(\neg\Diamond A\vee\Diamond B)\vee(\Diamond A\vee\neg\Diamond B)}$$

The second option is to apply Strong Kleene conjunction at the assertion level, i.e. $\mathbf{pex}^{IE}\ \phi = \frac{\phi}{\neg\psi_1}\blacktriangle\frac{\phi}{\neg\psi_2}\blacktriangle\cdots$ for exclusions $\psi_1, \psi_2, \ldots$. This application of $\blacktriangle$ minimizes the exclusion, and therefore seems intuitively preferable given the Strong Kleene approach that seeks to minimize presuppositions. The following computation shows this option yields the correct homogeneity presupposition:

$$\mathbf{pex}^{IE}_{C_2}\ \mathbf{pex}^{IE}_{C_1}\ \ \Diamond(A \vee B) = \frac{\Diamond(A \vee B)}{\neg \frac{\Diamond A}{\neg \Diamond B}} \blacktriangle \frac{\Diamond(A \vee B)}{\neg \frac{\Diamond B}{\neg \Diamond A}} \tag{17}$$

$$= \frac{\Diamond(A \vee B)}{\Diamond B \rightarrow \Diamond A} \blacktriangle \frac{\Diamond(A \vee B)}{\Diamond A \rightarrow \Diamond B} \tag{18}$$

$$= \frac{\Diamond(A \vee B)}{(\Diamond B \leftrightarrow \Diamond A) \vee (\Diamond A \wedge \neg \Diamond(A \vee B)) \vee (\Diamond B \wedge \neg \Diamond(A \vee B))} \tag{19}$$

$$= \frac{\Diamond(A \vee B)}{\Diamond B \leftrightarrow \Diamond A} \tag{20}$$

2.5 Negative Free Choice

The Strong Kleene approach assumes a classical account of the Boolean operators, and therefore the equivalence $\neg\Box(A \wedge B) \equiv \Diamond(\neg A \vee \neg B)$ holds. The right hand side of the equivalence has the same structure as the positive FC case with $\neg A$ and $\neg B$ instead of A and B. But to be sure that the above computation for the positive case applies, we need to verify that the right sets of scalar alternatives are available when $\mathbf{pex}^{IE}$ recursively applies in the following:

$$\mathbf{pex}^{IE}_{C'_2}\ \mathbf{pex}^{IE}_{C'_1}\ \ \neg\Box(A \wedge B) \tag{21}$$

We show now that C'_1 and C'_2 each contain elements that are equivalent to any term that can be derived from an element of C_1 of C_2 in the positive case, by replacing A with $\neg A$ and B with $\neg B$. When we have shown that, and can also see that no other alternatives that are generated interfere, it implies that the computation of the positive case goes through for the negative case too.

For the inner $\mathbf{pex}^{IE}$, C'_1 contains $\neg\Box A$, $\neg\Box B$, and $\neg\Box(A \wedge B)$. But these three alternatives in C'_1 are equivalent to the result of $\Diamond\neg A$, $\Diamond\neg B$, and $\Diamond(\neg A \vee \neg B)$. In duality with the positive case, we put aside disjunctive alternative $\neg\Box(A \vee B)$ and alternatives weaker than the scope of $\mathbf{pex}^{IE}$. Finally C_1 may contain alternatives derived by deleting $\neg$ such as $\Diamond(A \wedge B)$, $\Diamond A$, and $\Diamond B$, but these and their negation are contingent with negative free choice inference $\neg\Box(A \wedge B)$. Therefore the presence of these alternatives does not interfere with the derivation of negative free choice, and we can also put these aside.

For the outer $\mathbf{pex}^{IE}$, C'_2 contains at least these two elements which are again equivalent to elements of C_2 after replacing of A with $\neg A$ and B with $\neg B$:

$$\mathbf{pex}^{IE}_{C_1} \neg\Box A = \frac{\neg\Box A}{\neg\neg\Box B} = \frac{\Diamond\neg A}{\neg\Diamond\neg B} \tag{22}$$

$$\mathbf{pex}^{IE}_{C_1} \neg\Box B = \frac{\neg\Box A}{\neg\neg\Box A} = \frac{\Diamond \neq B}{\neg\Diamond\neg A} \tag{23}$$

Therefore the basic case of negative free choice follows from the validity of De Morgan's Law and the positive free choice case, as the following shows:

(24) $$\mathbf{pex}^{IE}_{C_2'}\ \mathbf{pex}^{IE}_{C_1'}\ \neg\Box(A\wedge B) = \mathbf{pex}^{IE}_{C_2'}\ \mathbf{pex}^{IE}_{C_1'}\ (\Diamond\neg A)\vee(\Diamond\neg B)$$

(25) $$= \frac{\Diamond(\neg A\vee\neg B)}{\Diamond\neg B\leftrightarrow\Diamond\neg A}$$

(26) $$= \frac{\neg\Box(A\wedge B)}{\neg\Box A\leftrightarrow\neg\Box B}$$

Taken together the assertion and presupposition entails $\neg\Box A$ and $\neg\Box B$, which is the negative FC inference.

Summing up, in this section we have shown that positive and negative FC can be derived using recursive $\mathbf{pex}^{IE}$, without invoking insertion of an intermediate local accommodation operator. In the course of the derivation, we discovered two novel aspects of how presuppositional exhaustification and a Strong Kleene semantics interact that may be of more general interest. First, we observed in (7) that presupposing a presuppositional statement $\frac{a}{p}$ could be taken to yield either a weak presupposition that is only violated if p is false, or a stronger one that is also violated if p is not true. The weak option is more in line with the Strong Kleene evaluation, and is the only option consistent with the results in this section. Secondly, in a Strong Kleene system presuppositional exclusion of multiple alternatives could be described in two ways. The stronger option is to conjoin the negated alternatives and then add the result as a presupposition to the assertion. The weaker option is to first add each negated alternative as a presupposition to the assertion, and then Strawson conjoin the results obtained. We showed in (11) and (17) that the results of the options can differ, and that, for the result in this section to stand, the weaker option must be available.

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