Mixed-polarity pluralities: a solution to van Benthem's problem[1]

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Abstract. In this paper, I suggest a new way of resolving a number of problems with an orthodox treatment of modified numerals as existential quantifiers over pluralities. It's well known that existential quantification renders upper bounds inert—an issue known as 'van Benthem's problem'. This can be addressed by building a notion of maximality into the semantics of modified numerals, but this fails to capture attested cumulative readings due to how maximality takes scope with respect to other operators. Existing work addressing this compositional stalemate exploits powerful mechanisms for side-stepping scopal interactions, such as post-suppositions (e.g., Brasoveanu 2013). Here, I maintain a simple treatment of modified numerals as existential quantifiers over pluralities, while significantly enriching the ontology of pluralities themselves. Concretely, I explore the idea that pluralities can encode both positive and negative information, building on recent work by Bledin (2024).

Keywords: quantification, numerals, maximality, cumulativity, plurality, negation.

1. Background

1.1. van Benthem's problem

Modified numerals such as *exactly three* can be analyzed as determiners within GQ (Generalized Quantifier) theory (Mostowski, 1957; Barwise and Cooper, 1981, a.o.).

(1) $\text{exactly } 3(A,B) \iff \#(A\cap B)=3$

GQ-theory however elides any distinction between singular and plural determiners. A DP of the form 'exactly *n* NP' is semantically plural, as diagnosed by its compatibility with collective predication.

(2) Exactly 10 soldiers surround the castle. cf. #Napoleon surrounds the castle.

It is therefore tempting to analyze, e.g., "exactly *n* soldiers" as an existential quantifier over pluralities consisting of *n* atomic *soldier* individuals.[2] This view however predicts truth conditions that are too weak for distributive predicates, as illustrated by (3).[3]

(3) a. Exactly 3 soldiers sneezed.
b. $\exists X, X \text{ are soliders}, \#X=3, \forall x \leq_{At} X, x \text{ sneezed}$ *unattested*

To see the problem, imagine a scenario in which there is a plurality of soldiers $a\oplus b\oplus c\oplus d$, each of whom sneezed. Since, a,b,c each sneezed, it follows that there is a group $X'\leq X$ (i.e., $a\oplus b\oplus c$), where $\#X'=3$, and each individual in X' sneezed. (3a) is therefore expected

[1]I'd especially like to thank Justin Bledin for extremely helpful feedback on an earlier version of this paper. I'm also grateful to Amir Anvari, Andrew Kato, and Filipe Hisao Kobayashi for much useful discussion surrounding this topic, as well as to participants of Sinn und Bedeutung 29, in Sicily. Any remaining mistakes are my own.

[2]I assume that the cardinality function #, when applied to an individual, returns the cardinality of the set of distinct atomic parts. $\leq$ is used to indicate plural-parthood, familiar from the work of Link (1983). $\leq_{At}$ is reserved for *atomic* plural parts, i.e., those which don't themselves have proper plural parts.

[3]Collective predication exhibits different behavior. Discussion of this point is deferred until section 6.1.

to be true in this scenario, but it is intuitively judged to be false. The problem here is that existential quantification renders the upper-bound of the modified numeral inert, and (3a) is therefore incorrectly predicted to be equivalent to "three soldiers sneezed". This dilemma and is commonly referred to as 'van Benthem's problem' (van Benthem 1986).

1.2. Maximality and cumulative readings

One way of reinstating the upper bound is given in (4b), which adds the condition that the *maximal* plurality of soldiers that sneezed has a cardinality of 3.

(4) a. Exactly 3 soldiers sneezed.

b. $\exists X, X \text{ are soldiers}, \#X = 3, \forall x \leq_{At} X, x \text{ sneezed}$
$\wedge \neg\exists X', X' \text{ are soldiers}, X < X', \forall x \leq_{At} X', x \text{ sneezed}$

The obvious next step is to build this maximality requirement directly into the semantics of "exactly *n*", as sketched in (5). This would be satisfactory for simple cases of distributive predication, but it runs into immediate issues with cumulative readings.

(5) $[\![\text{exactly } n \text{ soliders}]\!] = \lambda P\,.\,\exists X[X \text{ are soliders}, \#X = n, \forall x \leq_{At} X, P(x),$
$\neg\exists X', X' \text{ are soldiers}, X < X', \forall x' \leq_{At} X', P(x')]$

Alongside the expected doubly-distributive reading (discussed further in section 4.2), the sentence in (6) has a so-called cumulative reading (Scha, 1981; Krifka, 1999); it says, informally, that there were duels involving Spartans and Trojans, where there were exactly 2 Trojan-dueling Spartans, exactly 3 Spartan-dueling Trojans.

(6) Exactly 2 Spartans dueled exactly 3 Trojans.

As shown in detail by Brasoveanu (2013) and Charlow (2021), building maximality into the semantics of (5), together with standard assumptions about cumulative readings, gives rise to a weaker reading which Charlow calls 'pseudo-cumulative'. Allowing the maximality conditions to interact scopally predicts that (6) can mean that *the maximal group of Spartans who dueled 3 Trojans between them consists of* 2, as in (7).

(7) $\exists X[X \text{ are Spartans}, \#X = 2, X \text{ dueled ex. 3 Trojans}$ *unattested*
$\neg\exists X', X' \text{ are Spartans}, X < X', X' \text{ dueled ex. 3 Trojans}]$

Brasoveanu (2013) argues in detail (further reinforced by Charlow 2021) that the pseudo-cumulative reading is unavailable. The attested cumulative reading, paraphrased in (8), rather involves 'tallying-up' the Spartans who dueled Trojans, and the Trojans dueled with by Spartans — the two groups should have cardinalities of 2 and 3 respectively. In order to derive this reading compositionally, note that it isn't sufficient to exploit the 'non-maximal' entry for modified numerals in (3b), as van Benthem's problem arises in cumulative sentences too—to derive the attested cumulative reading in (8) the upper-bounds of the numerals should remain intact, and furthermore they shouldn't scopally interact (Landman, 2000).

(8) *Cumulative truth conditions (attested):*
$\#\{x \mid x \text{ a Spartan who dueled a Trojan}\} = 2$
$\wedge \#\{y \mid y \text{ a Trojan who was dueled with by a Spartan}\} = 3$

1.3. The existential entailment problem

There is an additional problem with treating numerals as existential quantifiers over pluralities. Consider, e.g., modified numerals of the form 'less than *n*', and the predicted truth conditions with a distributive predicate. Even ignoring the problem of the upper-bound, (9b) comes with an additional undesirable prediction—it entails that there *at least one boy sneezed*, given orthodox assumptions concerning the ontology of pluralities (Link, 1983). Nevertheless, (9a) is intuitively felt to be true in case no boys in fact sneezed. Following Buccola and Spector (2016), I'll refer to this as the 'existential entailment problem'.[4]

(9) a. Less than 3 boys sneezed.
b. $\exists X, X \text{ are boys}, \#X < 3, \forall x \leq_{At} X, x \text{ sneezed}$ *unattested*

1.4. Summary

To sum up, a treatment of modified numerals as existential quantifiers over pluralities leads to a range of problems: (i) existential quantification renders upper-bounds inert (van Benthem's problem), (ii) attempting to address van Benthem's problem by introducing a maximality component leads to the problem of cumulative readings, (iii) existentially quantifying over pluralities leads to an undesirable existential entailment. Interestingly, the classical GQ-theoretic treatment of modified numerals as determiners does not run into van Benthem's problem or the existential entailment problem, but it doesn't capture the fact that numeral expressions introduce pluralities, as evidenced by their compatibility with collective predications and cumulative readings. The current consensus in the literature is that these problems are nigh on impossible to resolve without introducing fairly heavy-duty compositional machinery for side-stepping scopal interactions between maximality conditions (Krifka, 1999; Brasoveanu, 2013; Charlow, 2021; Haslinger and Schmitt, 2020).

I'll pursue a unified approach to this constellation of properties by pursuing a different kind of strategy: namely, I conjecture that the source of the issues discussed here is the fact the orthodox mereological structure assumed for the domain of pluralities (Link, 1983) is not sufficiently expressive. In the remainder of this paper, I develop a fragment starting from a notion of plurality encoding both positive and negative information. The ultimate goal will be to show that this fragment supports a simple existential semantics for numeral expressions, as well as quantificational expressions more generally, while retaining advantages of GQ-theory. Resolutions to van Benthem's problem, and the existential entailment problem will follow immediately, while a natural generalization of cumulative readings will interact well with the resulting semantics for modified numerals.

2. The polarized fragment

I'll build primarily on an idea due to Bledin (2024) (see also Akiba 2009; Fine 2017a; Fine 2017b for important antecedent work), who introduces a distinction between *positive* and *neg-*

[4]To reinforce the point that an existential inference is undesirable, Buccola and Spector (2016) point out that 'less than *n*' licenses weak NPIs in its scope.

(1) Less than 3 boys have read any of these boys.

ative information in the domain of individuals. The idea, informally, is that for every atomic individual in the domain, e.g., Kit, there exists a ‘negative counterpart’, Kit^-; atomic individuals and their negative counterparts stand in a one-to-one relationship. The utility of negative counterparts will be that they may be true relative to a predicate P—concretely, Kit^- is true of P iff Kit is false of P.[5]

2.1. Polarizing the domain

This basic idea can be formalized as follows: following Bledin (2024), a *polarized* domain, $D^\pm$ is constructed from a base domain of atomic individuals D by: (i) associating each atomic individual $x \in D$ with a unique negative counterpart $x^- \in D^\pm$, (ii) closing $D^\pm$ under mereological sum-formation, (iii) filtering out the incoherent pluralities (Akiba, 2009).[6]

The polarized domain $D^\pm$ contains ordinary atomic individuals drawn from D, as well as *negative counterparts*, which stand in a one-to-one relationship with atomic individuals drawn from D. An atomic individual is *negative* if it is in $D^\pm - D$, and *positive* if it is in D. For convenience, I’ll use x^+ as a metavariable over ordinary atomic individuals, and x^- as a metavariable over negative atomic individuals, s.t., x^- is the unique negative counterpart of x^+.[7]

I’ll furthermore assume that the polarized domain is *closed under mereological sum-formation* (Link, 1983). This means that the polarized domain contains, e.g., wholly-positive pluralities such as $a^+ \oplus b^+$, wholly-negative pluralities such as $a^- \oplus b^-$, as well as mixed-polarity pluralities such as $a^+ \oplus b^-$. There’s an important proviso ‘incoherent’ pluralities, such as $a^+ \oplus a^-$, are filtered out; a plurality is considered incoherent just in case it contains both an individual x as well as x’s negative counterpart (10).

(10) A plurality $X \in D^\pm$ is *incoherent* if $\exists x \in D[x^+ \leq_{At} X \wedge x^- \leq_{At} X]$

At this juncture, it’s important to note that the resulting structure, unlike a classical mereology, will contain $n \geq 1$ maximal elements with respect to mereological parthood, due to removal of incoherent pluralities. Given a base domain (11a), the maximal elements of the resulting polarized domain are shown in (11b).[8]

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5 The term ‘negative counterpart’ is not to be confused with Lewis’s (1968) counterpart theory; Bledin (2024) uses the term ‘orthogonal counterpart’.

6 The fragment developed in this section is inspired particularly by the work of Bledin (2024), but departs from his work in many non-trivial ways. For example, Bledin effectively assumes that the base domain is closed under sum-formation, before adding in negative counterparts, and closing under sum-formation for a second time. This allows in additional entities, such as *negative pluralities*, e.g., $(a \oplus b \oplus c)^-$. Additionally, in Bledin’s fragment incoherent pluralities are not filtered out at this stage in the construction of the polarized domain. As far as I’m able to determine, the additional structure afforded in Bledin’s fragment isn’t necessary for the goals of the present work. Moreover, Bledin develops a *truthmaker semantics*, where the positive/negative distinction is extended to verifying/falsifying situations, whereas the fragment developed here is a simple extensional one. I defer a detailed comparison with Bledin’s particular implementation to future work.

7 Worries concerning the ontological status of ‘negative individuals’ may be assuaged by considering different ways of formalizing this idea giving rise to an isomorphic structure, for example, polarized individuals can be modeled as pairs of ordinary individuals and boolean values, i.e., $a^+ \approx (a,1)$ and $a^- \approx (a,0)$. See Bledin, 2024 for additional discussion on this point.

8 The definition of the *maximality* operator is as follows; it takes a set of elements ordered by $\leq$, and returns the maximal elements with respect to $\leq$ (see, e.g., Winter 2001, p. 152).

(1) $\mathsf{Max}_{\leq}(P) = \{X \in P \mid \forall X' \in P, X \leq X' \rightarrow X = X'\}$

(11) a. $D = \{a,b,c\}$

b. $$\mathsf{Max}_{\leq}(D^{\pm}) = \left\{ \begin{array}{c} a^+ \oplus b^+ \oplus c^+, \\ a^+ \oplus b^+ \oplus c^-, a^+ \oplus b^- \oplus c^+, a^- \oplus b^+ \oplus c^+, \\ a^+ \oplus b^- \oplus c^-, a^- \oplus b^+ \oplus c^-, a^- \oplus b^- \oplus c^+, \\ a^- \oplus b^- \oplus c^- \end{array} \right\}$$

I'll close this section by introducing a notational convention that will be frequently exploited throughout the rest of this paper: given a plurality X, I'll use X^+ to refer to the atomic parts that are *positive*, and X^- to refer to positive individuals whose negative counterparts are atomic parts of X. I.e., $(a^+ \oplus b^+ \oplus c^-)^+ = \{a,b\}$, and $(a^+ \oplus b^+ \oplus c^-)^- = \{c\}$.

(12) $X^+ := \{x^+ \mid x^+ \leq_{At} X\} \qquad X^- := \{x^+ \mid x^- \leq_{At} X\}$

2.2. Polarized distributivity

For ease of exposition, I initially present a fragment with only distributive predicates, which I assume are only defined for positive atomic individuals. In the polarized fragment, composing an element of $D^{\pm}$ with a distributive predicate is mediated by a distributivity operator Dist, which given a plurality X, and a predicate P, asserts that the positive atoms are true of P, and the positive atoms with negative counterparts in P are false of P (13)—see (14) for an illustration.

(13) $\mathsf{Dist}(P) := \lambda X \in D^{\pm}.\forall x \in X^+, P(x) = 1 \wedge \forall x \in X^-, P(x) = 0$

(14) $\mathsf{Dist}(\text{sneezed})(a^+ \oplus b^+ \oplus c^-) \iff a$ sneezed $\wedge\, b$ sneezed $\wedge\, b$ didn't sneeze

2.3. Plurality and maximality

I assume that proper names and singular definite descriptions exclusively denote positive individuals; in the polarized fragment, negative individuals only become relevant once the semantics of plural NPs is taken into account.[9] As in Link's (1983) influential account, I assume that the denotation of a plural NP crucially exploits the fact that the domain is closed under mereological sum-formation. However, I will stipulate that the denotation of a plural NP only includes *maximal* pluralities, relative to the meaning of the NP. This will pay dividends once I move to the semantics of quantificational expressions.[10]

(15) $[\![\text{boys}]\!] = \mathsf{Max}_{\leq}\{X \in D^{\pm} \mid \forall x \in X^+ \cup X^-, x \text{ is a boy}\}$

For concreteness, assuming that the base domain D contains boys $b_{1\ldots3}$, the denotation of "boys" will only include pluralities which contain, for each $b \in \{b_1,b_2,b_3\}$, either b or b's negative counterpart, e.g., $b_1^+ \oplus b_2^+ \oplus b_3^-$ but not $b_1^+ \oplus b_2^-$. Due to the ban on incoherent sums, there are many such maximal *boy*-pluralities. A helpful intuition regarding maximal *boy* pluralities is that they specify, for each boy, whether or not he has some yet-to-be-specified property.

[9]But see Bledin (2024), who argues partially on the basis of collective conjunctions that natural language negation can introduce a negative individual.

[10]I'm grateful to Filipe Hisao Kobayashi (p.c.) for suggesting this particular formulation.

3. Determiners as predicates of pluralities

3.1. *Some* and *all*

Recall that in the polarized fragment, composition of pluralities with distributive predicates must be mediated by the polarized distributivity operator. Quantificational determiners can be defined directly as predicates of such pluralities. Starting from the simplest cases, the denotations of "some" and "all" are given below.

(16) $[\![\text{some}]\!] = \{X \mid X^+ \neq \emptyset\} \qquad [\![\text{all}]\!] = \{X \mid X^- = \emptyset\}$

Since quantificational determiners, like plural NPs, denote predicates of pluralities, they may compose as intersective modifiers (e.g., via Heim and Kratzer's 1998 rule of *Predicate Modification*). In order to understand how this semantics operates, it will be useful to start by considering a concrete domain of *boys* and the corresponding maximal *boy*-pluralities. For a base domain with boys $b_{1\ldots 3}$, this is given in (17).

(17) $$[\![\text{boys}]\!] = \left\{ \begin{array}{c} b_1^+ \oplus b_2^+ \oplus b_3^+, \\ b_1^+ \oplus b_2^+ \oplus b_3^-, b_1^+ \oplus b_2^- \oplus b_3^+, b_1^- \oplus b_2^+ \oplus b_3^+, \\ b_1^+ \oplus b_2^- \oplus b_3^-, b_1^- \oplus b_2^+ \oplus b_3^-, b_1^- \oplus b_2^- \oplus b_3^+, \\ b_1^- \oplus b_2^- \oplus b_3^- \end{array} \right\}$$

Starting with *some*, if we intersectively modify *boys*, this will simply amount to eliminating the wholly-negative maximal plurality:

(18) $$[\![\text{some}]\!] \cap [\![\text{boys}]\!] = \left\{ \begin{array}{c} b_1^+ \oplus b_2^+ \oplus b_3^+, \\ b_1^+ \oplus b_2^+ \oplus b_3^-, b_1^+ \oplus b_2^- \oplus b_3^+, b_1^- \oplus b_2^+ \oplus b_3^+, \\ b_1^+ \oplus b_2^- \oplus b_3^-, b_1^- \oplus b_2^+ \oplus b_3^-, b_1^- \oplus b_2^- \oplus b_3^+, \\ \xcancel{b_1^- \oplus b_2^- \oplus b_3^-} \end{array} \right\}$$

Moving on to *all*, if we intersectively modify *boys*, it will eliminate any plurality with any negative parts. This will eliminate all but one plurality.

(19) $$[\![\text{all}]\!] \cap [\![\text{boys}]\!] = \left\{ \begin{array}{c} b_1^+ \oplus b_2^+ \oplus b_3^+, \\ \xcancel{b_1^+ \oplus b_2^+ \oplus b_3^-}, \xcancel{b_1^+ \oplus b_2^- \oplus b_3^+}, \xcancel{b_1^- \oplus b_2^+ \oplus b_3^+}, \\ \xcancel{b_1^+ \oplus b_2^- \oplus b_3^-}, \xcancel{b_1^- \oplus b_2^+ \oplus b_3^-}, \xcancel{b_1^- \oplus b_2^- \oplus b_3^+}, \\ \xcancel{b_1^- \oplus b_2^- \oplus b_3^-} \end{array} \right\}$$

In order to compose the resulting sets of pluralities with a scope, I'll invoke a covert operation of *existential raising* (ER) (Winter, 2001), which is defined in (20); it takes a predicate of pluralities P and asserts that one such plurality is true of the scope Q. Composition with a distributive scope is mediated by the polarized distributivity operator. A quantificational statement will thereby assert that there is a member of the DP denotation that is distributively true of the scope. An schematic Logical Form for the sentence "some boys sneezed" is given in (21).

(20) $\mathsf{ER} := \lambda P \,.\, \lambda Q \,.\, \exists X \in P, Q(X) = 1$

(21) *true* iff $\exists X \in [\![\text{boys}]\!], X^+ \neq \emptyset, \forall x \in X^+, x$ sneezed, $\forall x \in X^-, x$ didn't sneeze

- $\lambda Q . \exists X, X \in [\![\text{boys}]\!], X^+ \neq \emptyset, Q(X) = 1$
  - ER
  - $\{X \mid X \in [\![\text{boys}]\!], X^+ \neq \emptyset\}$
    - some
    - boys
- $\lambda X . \forall x \in X^+, x$ sneezed, $\forall x \in X^-, x$ didn't sneeze
  - Dist
  - sneezed

It's easy to see, given (21) that the resulting truth conditions will be existential. (21) says that there is some 'complete specification' of *boys-sneezing*, s.t., in that specification one boy did in fact sneeze. In other words, existentially raising "some boys" amounts to a generalized disjunction of statements such as b_1 *sneezed,* b_2 *sneezed, and* b_3 *didn't sneeze*, etc.[11]

For *all*, it's even easier to see that ER delivers the correct universal truth conditions. This is essentially because there is always at most one maximal plurality with no negative parts, so ER is tantamount to just distributively predicating the scope of this plurality. The resulting truth conditions are conjunctive.

(22) $ER([\![\text{all}]\!] \cap [\![\text{boys}]\!])(\mathsf{Dist}(\text{sneezed})) = \mathsf{Dist}(\text{sneezed})(b_1^+ \oplus b_2^+ \oplus b_3^+)$

The semantics sketched here can be easily extended to *no* and *not all* in the obvious way. Verifying the correctness of the denotations below is left as an exercise for the reader.

(23) $[\![\text{no}]\!] = \{X \mid X^+ = \emptyset\}$ $\quad$ $[\![\text{not all}]\!] = \{X \mid X^- \neq \emptyset\}$

3.2. Numerals

The compositional regime assumed here treats determiners as predicates, and invokes both existential raising and a distributivity operator in order to compose DP denotations with their scopes. This parallels a widespread proposal for composing numerals (see, e.g., Winter 2001), and so unsurprisingly these assumptions can be extended straightforwardly to numeral expressions, which also receive a straightforward treatment as predicates of pluralities. Whereas *some* and *all* place constrains on X^+ and X^- respectively, numerals uniformly place cardinality constraints on X^+. For reasons that will become immediately apparent, I'll assume that bare numerals have a basic *at least* semantics.

(24) $[\![\text{two}]\!] = \{X \mid \#X^+ \geq 2\}$

The bare numeral 'two' is true of pluralities that have *at least* 2 positive parts. When applied to, e.g., the maximal *boy* pluralities as an intersective modifier, this will include, e.g., $b_1^+ \oplus b_2^+ \oplus b_3^+$ and $b_1^+ \oplus b_2^+ \oplus b_3^-$, but exclude, e.g., $b_1^+ \oplus b_2^- \oplus b_3^-$. Via existential raising, "two boys sneezed" will amount to a generalized disjunction of statements which specify, for each boy, whether or not he sneezed, where each statement says that at least two boys sneezed.

[11]Bledin (2024) and Akiba (2009) make closely related proposals concerning how quantificational expressions can be modeled as predicates of structured entities. The new contribution here is an explicit, decompositional semantics for determiners which treats them as intersective modifiers.

(25) $\mathsf{ER}([\![2]\!] \cap [\![\text{boys}]\!])(\mathsf{Dist}(\text{sneezed})) \iff \#\{x \mid x \text{ is a boy and } x \text{ sneezed}\} \geq 2$

Turning now to modified numerals of the form *exactly n*, without committing to a detailed decompositional treatment, I'll assume that they convey an *exactly* semantics, in contrast to bare numerals. Crucially, this will mean that, e.g., $b_1^+ \oplus b_2^+ \oplus b_3^+$ will be included in the denotation of "two boys", but will be excluded from the denotation of "exactly two boys" (since there are more than two positive parts).

(26) $[\![\text{exactly 2}]\!] = \{X \mid \#X^+ = 2\}$

(27) $\mathsf{ER}([\![\text{exactly 2}]\!] \cap [\![\text{boys}]\!])(\mathsf{Dist}(\text{sneezed})) \iff \#\{x \mid x \text{ is a boy and } x \text{ sneezed}\} = 2$

3.3. Existential entailments and van Benthem's problem

In the polarized fragment, both van Benthem's problem and the existential entailment problem are immediately addressed. To see why, consider the entry for 'less than 3', and what happens when we intersectively modify the set of maximal *boy* pluralities. Crucially, the wholly-positive maximal *boy*-plurality is eliminated—in other words, the upper-bound is not nullified by existential raising, since it is already inherent in the NP denotation. Moreover, the resulting DP-denotation predicts that "less than 3 boys sneezed" can be true if, e.g., no boys sneezed, since the wholly-negative plurality is included. This resolves the existential entailment problem, correctly capturing the fact that modified numerals of the form "less than *n*" are monotone decreasing.

(28) $[\![\text{less than 3}]\!] = \{X \mid \#X^+ < 3\}$

(29) $$[\![\text{less than 3}]\!] \cap [\![\text{boys}]\!] = \left\{ \begin{array}{c} \cancel{b_1^+ \oplus b_2^+ \oplus b_3^+}, \\ b_1^+ \oplus b_2^+ \oplus b_3^-, b_1^+ \oplus b_2^- \oplus b_3^+, b_1^- \oplus b_2^+ \oplus b_3^+, \\ b_1^+ \oplus b_2^- \oplus b_3^-, b_1^- \oplus b_2^+ \oplus b_3^-, b_1^- \oplus b_2^- \oplus b_3^+, \\ b_1^- \oplus b_2^- \oplus b_3^- \end{array} \right\}$$

A more extreme version of the existential entailment problem is manifested by the numeral 'zero'; statements involving 'zero' seem truth-conditionally equivalent to negative existential statements; (30) is incompatible with the existence of any boy who sneezed (Landman, 2004).

(30) Zero boys sneezed. $\Rightarrow$ *No boys sneezed*

Similarly to the case of 'less than *n*', this can be immediately resolved by the current treatment of numerals; 'zero' is simply true of pluralities with exactly zero positive parts. Since there can be at most one such maximal *boy*-plurality with zero positive parts, the resulting truth conditions for "zero boys sneezed" amount to asserting that the wholly negative maximal *boy* plurality distributively sneezed, i.e., that no boys sneezed.

(31) $[\![\text{zero}]\!] = \{X \mid \#X^+ = 0\}$

There are some outstanding issues regarding the semantics of *zero* specifically that I will pick back up in section 5.1.

3.4. Other determiners and conservativity

Conservativity is a formal property that picks out a subset of GQ-theoretic determiners, which famously has been claimed to universally hold of determiner denotations in natural language (Keenan and Stavi, 1986). The definition of conservativity is given in (32); informally, a determiner is conservative just in case elements in the scope A not also in the restrictor B never bear on the truth of a statement of the form $D(A,B)$.

(32) A GQ-theoretic determiner D is *conservative* iff $D(A,B) \iff D(A,A \cap B)$

Once negative individuals are taken into account, properties of maximal pluralities are expressive enough to capture any *conservative* GQ-theoretic determiner. Consider, for example, the case of proportional *most*; the classic GQ-theoretic treatment of *most* is given in (33). Given a restrictor A, and a scope B, it says that the number of A-elements that are B-elements is greater than the number of A-elements that aren't.

(33) $\mathsf{most}_{GQ}(A,B) := \#(A \cap B) > \#(A - B)$

This is mirrored in the entry for *most* as a predicate of pluralities, given in (33). Intuitively, X^+ corresponds to the elements of the restrictor that are also in the scope, and X^- corresponds to the elements of the restrictor that aren't in the scope. It can easily be appreciated that the truth conditions predicted by the entry in (34) are equivalent to those predicted by (33) for a sentence such as "most boys sneezed". The entry in (34) eliminates all but the maximal *boy*-pluralities where the positive boys outweigh the negative boys. Such a plurality is distributively true of *sneezed* iff the number of boys who sneezed is greater than the boys who didn't sneeze.

(34) $[\![\text{most}]\!] = \{X \mid \#X^+ > \#X^-\}$

There is a general recipe that allows us to translate from a conservative GQ-theoretic determiner to a predicate of pluralities D_{Pred}. Let D_{Cons} be an arbitrary conservative determiner. The mapping is given in (35).

(35) $D_{Pred} := \{X \in D^\pm \mid D_{Cons}(X^+ \cup X^-, X^+)\}$

Consider how this works in the case of *most*: (36) shows how the recipe in (35) applied to most_{GQ} derives the entry in (34) via some routine equivalences.

(36)
$$\begin{aligned} [\![\text{most}]\!] &= \{X \in D^\pm \mid \mathsf{most}_{GQ}(X^+ \cup X^-, X^+)\} \\ &= \{X \in D^\pm \mid \#((X^+ \cup X^-) \cap X^+) > \#((X^+ \cup X^-) - X^+)\} \\ &= \{X \in D^\pm \mid \#X^+ > \#X^-\} \end{aligned}$$

This recipe only works for conservative determiners. To illustrate this, consider the non-conservative GQ-theoretic determiner I, which checks whether the cardinalities of the restrictor and the scope are identical. Applying the recipe in (35) to I results in the predicate of pluralities given in (38). Consider however what (38) does—it will compare the cardinality of the set of *all elements of the restrictor* ($X^+ \cup X^-$), with elements in the restrictor that are true of the scope (X^+), so the truth conditions predicted by (38) are distinct from those of (37). Tellingly, (38) instead characterizes a hypothetical $I'(A,B)$, which holds iff $\#A = \#(A \cap B)$—note that this is equivalent to *all*, a conservative determiner!

(37) $I(A,B) \iff \#A = \#B$

(38) $I_{Pred} = \{X \in D^{\pm} \mid \#(X^{+} \cup X^{-}) = \#X^{+}\}$

This is not an accident, because in fact properties of pluralities in the current setting may *only* express conservative determiners. Intuitively, this is because the determiner always picks out a subset of the set of maximal pluralities denoted by the restrictor; each maximal plurality contains information about only the elements of the restrictor true of the scope (X^{+}) and the elements of the restrictor false of the scope (X^{-}). There is simply no way of making reference to elements of the scope false of the restrictor. The claim here, then, is that determiners as properties of maximal pluralities characterize the *conservative* GQ-theoretic determiners. I defer a full proof, and a more careful consideration of the logical properties of the system, to future work.

4. Cumulative readings

4.1. The doubly-distributive reading

Based on the assumptions outlined so far, it's already possible to derive an attested doubly-distributive reading for sentences with modified numerals, such as (39a). This is accomplished by existentially raising each quantifier, scoping them, and proceeding with composition via the polarized distributivity operator. The doubly-distributive reading of (39a) can be paraphrased as follows: *there's a group of exactly two boys X, s.t., each $x \in X$ ate exactly three pizza slices, and no larger group of boys each ate exactly three pizza slices.*. The Logical Form that derives these truth conditions is schematized in (39b), and parallels an established strategy for deriving doubly-distributive readings (see, e.g., Winter 2001).

(39) a. Exactly two boys ate exactly three pizza slices.

b. *Doubly-distributive Logical Form:*
$\exists X \in [\![\text{ex. 2 boys}]\!]\,(\mathsf{Dist}(\lambda x\,.\,\exists Y \in [\![\text{ex. 3 slices}]\!]\,(\mathsf{Dist}(\lambda y\,.\,x \text{ ate } y))(Y)))(X)$

4.2. The polarized cumulativity operator

In order to account for cumulative readings more generally, I'll follow Beck and Sauerland (2000) in positing a phrasal cumulativity operator. The definition of Beck and Sauerland's operator, which applies to a (curried) relation between individuals R, and returns a cumulative relation between pluralities, is given in (40):

(40) $R^{**}(X)(Y) \iff \forall y \leq_{At} Y, \exists x \leq_{At} X, R(x)(y) \land \forall x \leq_{At} X, \exists y \leq_{At} Y, R(x)(y)$

In providing a polarized definition of this operator, the intuition I'll aim to formalize is that, given two pluralities Y and X, R cumulatively holds of Y^{+} and X^{+} ('positive participation'), whereas nothing in Y^{-} holds of R with any of X, and nothing in X^{-} holds of R with any of Y ('negative participation'). This is formalized in (41). Note especially that the positive participation clause encodes the contribution Beck and Sauerland's operator.

(41) $\forall X,Y \in D^{\pm}, \mathsf{Cuml}(R)(X)(Y) \iff$

$$\overbrace{\forall y \in Y^+, \exists x \in X^+, R(x)(y), \forall x \in X^+, \exists y \in Y^+, R(x)(y)}^{\text{positive participation}}$$
$$\wedge \underbrace{\neg\exists(y,x), y \in Y^-, x \in (X^+ \cup X^-), R(x)(y), \neg\exists(y,x), x \in X^-, y \in (Y^+ \cup Y^-), R(x)(y)}_{\text{negative participation}}$$

To warm-up, consider, (42), which corresponds to the proposition that *boys 1 and 2 but not boy 3 (cumulatively) ate slices 1, 2, and 3 but not slice 4*. Informally, it guarantees that boys 1 and 2 cumulatively ate slices 1,2 and 3 (the same requirement delivered by Beck and Sauerland's operator), and furthermore that boy 3 didn't eat any slices, and slice 4 wasn't eaten by any boy. As we'll see, this polarized definition will be essential in delivering cumulative readings with modified numerals.

(42) $\mathsf{Cuml}(\mathsf{ate})(s_1^+ \oplus s_2^+ \oplus s_3^+ \oplus s_4^-)(b_1^+ \oplus b_2^+ \oplus b_3^-) \iff$

$$\overbrace{\forall b \in \{b_1, b_2\}, \exists s \in \{s_1, s_2, s_3\}, \mathsf{ate}(s)(b), \forall s \in \{s_1, s_2, s_3\}, \exists b \in \{b_1, b_2\}, \mathsf{ate}(s)(b)}^{\text{positive participation}}$$
$$\wedge \underbrace{\neg\exists s \in \{s_{1\ldots4}\}, \mathsf{ate}(s)(b_3), \neg\exists b \in \{b_{1\ldots3}\}, \mathsf{ate}(s_4, b)}_{\text{negative participation}}$$

4.2.1. Cumulative readings with modified numerals

Now we're finally in a position to demonstrate how the semantics outlined here for modified numerals interacts with *polarized* cumulativity. It will be useful to go through a concrete case. Assume boys $b_{1\ldots3}$ and slices $s_{1\ldots3}$, and consider what is predicted for the cumulative reading of the following sentence:

(43) Exactly two boys ate exactly two pizza slices.

Since the quantificational DPs are existentially raised, they take scope, but since they don't take *distributive* scope, they scopally commute. For the sake of exposition I'll assume a Logical Form where the DPs take surface scope, as in (44).

(44) $\exists X \in [\![\text{ex. 2 boys}]\!], \exists Y \in [\![\text{ex. 2 slices}]\!], \mathsf{Cuml}(\mathsf{ate})(Y)(X)$

As we've seen, both *exactly 2 boys* and *exactly 2 slices* range over maximal pluralities with exactly 2 positive atoms. As a result, (44) conveys the following:

(45) $$\exists X \in \left\{ \begin{array}{c} b_1^+ \oplus b_2^+ \oplus b_3^-, \\ b_1^+ \oplus b_2^- \oplus b_3^+, \\ b_1^- \oplus b_2^+ \oplus b_3^+ \end{array} \right\}, \exists Y \in \left\{ \begin{array}{c} s_1^+ \oplus s_2^+ \oplus s_3^-, \\ s_1^+ \oplus s_2^- \oplus s_3^+, \\ s_1^- \oplus s_2^+ \oplus s_3^+ \end{array} \right\}, \mathsf{Cuml}(\mathsf{ate})(Y)(X)$$

In short, it should be the case that the maximal group of boys who ate slices consists of just two boys, and the maximal group of slices that were eaten by boys consists of just two slices; no other boys ate any slices, and no other slices were eaten by any boys. These truth conditions correspond to the attested cumulative reading of (43).

4.3. Cumulative readings more generally

The mechanisms responsible for cumulative readings with upper-bounded modified numerals are very general, and extend to other numerals, as well as other quantificational expressions. A pertinent empirical observation is that the numeral *zero* remarkably supports what can be characterized as a 'cumulative reading' (cf. Landman 2004). In order to demonstrate this, consider the following context: a college professor has asked their students to submit questions in advance of their weekly seminar, and each student is allowed to submit more than one question. The professor asks their teaching assistant how many questions were submitted (fearing the worst). Their teaching assistant responds:

(46) Unfortunately, zero students submitted zero questions this week.

I contend that (46) can be naturally used to convey that *no student submitted any question this week*.[12] This is quite surprising, but it follows straightforwardly from a plural treatment of 'zero students', exploiting negative individuals, together with the polarized cumulativity operator. To understand how, consider again the entry assumed here for 'zero', repeated below as (47):

(47) $[\![\text{zero}]\!] = \{X \mid \#X^{+} = 0\}$

Due to the fact that plural nouns denote sets of *maximal* pluralities, both 'zero students' and 'zero questions' will denote singleton sets — namely, containing the wholly negative pluralities of *students* and *questions* respectively. The polarized cumulativity operator, will say of the subject that *no students submitted a question*, and will say of the object that *no question was submitted by a student* (note that the latter requirement is already entailed by the first). The Logical Form that gives rise to this reading is given in (48); it is the same Logical Form that affords cumulative readings elsewhere:

(48) $\exists X \in [\![\text{zero students}]\!], \exists Y \in [\![\text{zero questions}]\!], \mathsf{Cuml}(\mathsf{asked})(Y)(X)$
$\Rightarrow \mathsf{Cuml}(\mathsf{asked})(\iota\,\{Y \in [\![\text{questions}]\!] \mid Y^{+} = \emptyset\})(\iota\,\{X \in [\![\text{students}]\!] \mid X^{+} = \emptyset\})$
$\Rightarrow \neg\exists(x,y), x \text{ is a student}, y \text{ a question}, x \text{ asked } y$

To finalize this section, it's interesting to note that doubly-distributive readings with *zero* are contrary to cumulative reading. As previously, imagine that a college professor has asked

[12] Although most native English speakers I have consulted with agree with this judgment, a small number of consultants found this reading difficult to access. To reinforce the empirical claim, here are two naturally occurring cases found in online data. In each case, the context disambiguates in favor of the 'cumulative' reading:

(1) "Dear Reader, you need to unplug yourself because nobody outside Reddit cares or even knows about what arguments are going on there. I've met **zero** people in **zero** places in meat space that have heard of Rippetoe or Mehdi, [...] or any other God we cared to hold up..." (my emphasis)
`https://web.archive.org/web/20250212095935/https://purplespengler.blogspot.com/`

(2) "At the time btw **zero** people gave **zero** fucks about gain-of-function research. Trump cleared that in US between 2017 and 2020." (my emphasis)
`https://web.archive.org/web/20250212100005/https://pokerfraudalert.com/forum/showthread.php?19983-So-coronavirus-is-definitely-going-to-kill-a-few-of-us\%252Fpage638`

This data should of course be taken with a pinch of salt, since information about the authors' linguistic background is unavailable. Nevertheless, it is suggestive.

students to submit questions in advance. The professor is worried that some of the students may have forgotten, and asks their teaching assistant how many questions were submitted. The teaching assistant responds:

(49) Don't worry! Fortunately zero students submitted zero questions this week.

In this context (49) conveys that *every student submitted at least one question*. This follows from the Logical Form in (50), where both numerals take distributive scope. The details of the computation are elided, but are straightforward to reconstruct.

(50) $\exists X \in [\![\text{zero students}]\!]\,(\mathsf{Dist}(\lambda x\,.\,\exists Y \in [\![\text{zero questions}]\!]\,(\mathsf{Dist}(\lambda y\,.\,x \text{ asked } y))(Y)))(X)$

5. Other theories, and precedents

5.1. Pluralities and the bottom element

There are some precedents in the literature for enriching the ontology of pluralities to address a subset of the data considered in this paper. In order to give a semantics for the numeral *zero* as a predicate of pluralities, Bylinina and Nouwen (2018) suggest that the domain of pluralities constitutes what they describe as a full lattice, crucially including a unique bottom element $\bot$. Importantly, a pluralized predicate $P^{\times}$ always includes $\bot$ in the resulting denotation.

(51) $P^{\times} = \{\bot, a, b, c, a \oplus b, a \oplus c, b \oplus c, a \oplus b \oplus c\}$

The cardinality function # returns 0 for $\bot$, therefore their assumptions afford a simple predicative semantics for *zero* as follows:

(52) $[\![\text{zero}]\!]\,(X) \iff \#X = 0$

As they observe, this treatment, together with existential raising, gives rise to a special case of van Benthem's problem (see also Landman 2004). Namely, statements of the form "Zero *A* *B*" are expected to be tautological, since $A^{\times}$ and $B^{\times}$ always contain $\bot$. In order to resolve this issue, Bylinina and Nouwen, 2018 invoke obligatory exhaustification of *zero* relative to alternative numerals. The resulting meaning for *zero*-statements is given in (53); it's straightforward to verify that this is truth-conditionally equivalent to *no A B*.

(53) $\text{Zero } A\ B \iff \exists X, X \in A^{\times} \cap B^{\times}, \#X = 0, \neg\exists Y, Y \in A^{\times} \cap B^{\times}, \#Y > 0$

In comparing Bylinina and Nouwen's account to the present system, it's important to note that their analysis is only intended to capture the behavior of the numeral *zero* (and it is not clear how to extend their account to cumulative readings), whereas the present system has many more applications across the domain of numeral and quantificational expressions more generally. One interesting thing to note is that, in the present system, bare numerals generally are given a basic *at least* semantics, consider, e.g., (54a). This is because existential raising does not render upper-bounds inert. Nevertheless, I assumed that bare *zero* has a basic *exactly* semantics, as in (54b); with no upper-bound, using 'zero' in a sentence would otherwise be tautological. It might be considered dissatisfying to have a dis-parallel between the basic meaning of 'zero' and other numerals, in which case the present system is compatible with assuming a basic *at least* semantics for 'zero', and invoke obligatory exhaustification to derive (54b).[13]

[13] See, however Elliott, 2019; Kennedy, 2023 for arguments against this approach.

(54) a. $[\![\text{three}]\!] = \{\, X \in D^{\pm} \mid \#X \geq 3 \,\}$
b. $[\![\text{zero}]\!] = \{\, X \in D^{\pm} \mid \# = 0 \,\}$

Similarly, Buccola and Spector (2016) invoke the bottom element to address the *existential entailment problem* discussed in section 3.3. As discussed there, numerals of the form 'less than *n*', fail to carry an existential inference with distributive predicates. To resolve this, Buccola and Spector (2016) assume that distributive predicates, as well as plural nouns, have the bottom element in their denotations. Intuitively, wholly-negative pluralities in the present theory play a parallel role to the bottom element in Buccola and Spector (2016)'s account, grounded in a more general distinction between positive and negative individuals.

5.2. Other accounts of cumulative readings

Accounting for cumulative readings of modified numerals is something of a vexed issue. Many recent accounts take for granted Krifka's (1999) conclusion that an analysis of this kind of data in-line with standard compositional assumptions concerning scope is untenable. Brasoveanu (2013) for example exploits the expressive power of dynamic semantics to formalize the idea that the cardinality checks imposed by modified numerals are evaluated later than the maximality component of meaning—they constitute so-called 'post-suppositions'.[14]

Charlow (2021) explores two alternatives to Brasoveanu's account compatible with more orthodox assumptions regarding scope-taking. The first approach exploits *higher-order generalized quantifiers*; as Charlow notes, this account requires additional stipulations in order to rule out 'pseudo-cumulative' readings. The second approach relies update semantics (Heim 1982; Groenendijk, Stokhof, and Veltman 1996, etc.). Charlow (2021) argues in detail that, whichever account of cumulative readings of modified numerals is correct, it should rely on orthodox scope-taking mechanisms, due to the sensitivity of long-distance cumulative readings to islands. Consider, e.g., the following example from Charlow, 2021, p. 28—as he points out, this lacks a cumulative reading.

(55) Exactly three boys waved to a girl $[_{\text{island}}$ who owns exactly five donkeys].

The present account correctly captures the fact that cumulative readings with modified numerals are sensitive to restrictions on scope-taking—this is because, here, cumulative readings of modified numerals are just *ordinary cumulative readings*, no additional compositional mechanism is needed, beyond existentially raising the numeral expressions.

6. Open issues

6.1. Collective predication

I have not discussed how to integrate collective predication into the polarized fragment, but there is at least one obvious way of doing so. One may assume that collective predicates hold directly of mixed-polarity pluralities. Consider, e.g., the collective predicate *lift the piano*, and assume there are two distinct *lifting events*: one where $a \oplus b$ performed the lifting, and one where $a \oplus b \oplus c$ performed the lifting. The predicate would, by conjecture, have at least the following pluralities in its extension:

[14] See also Haslinger and Schmitt, 2020 for a bi-dimensional account that is reminiscent of Brasoveanu, 2013, but based on a different compositional regime. Haslinger and Schmitt's account is tailored to account for split-scope effects, which introduce significant complications which I won't have space to discuss here.

(56) $[\![\text{lift the piano}]\!] = \{a^+ \oplus b^+ \oplus c^-, a^+ \oplus b^+ \oplus c^+, \dots\}$

This would mean that a quantificational expression such as "exactly two people" could compose directly with the collective predicate.

(57) Exactly two people lifted the piano. $\exists X \in [\![\text{ex. 2 people}]\!], [\![\text{lifted the piano}]\!](X)$

In this scenario, the sentence is predicted to be true, since the group $a^+ \oplus b^+ \oplus c^-$ is in the extension of "exactly two people" (assuming the people are a, b, c). Note that the upper-bound of the modified numeral is rendered inert, since the sentence is still true, despite the fact that, separately, $a \oplus b \oplus c$ lifted the piano. According to Buccola and Spector (2016) this is a good prediction: they claim that once we turn to collective predicates the predicted non-upper-bounded truth conditions match speakers' intuitive judgments.

We can state a general recipe for moving from a classical denotation of a collective predicate P_{coll}, modeled as a predicate of sets of individuals, to its polarized counterpart as in (58). This says that a polarized plurality X is true of P_{coll} just in case either, (i) the set of positive parts is true of P_{coll}, or (ii) X is wholly negative, and no subset of its negative parts is true of P_{coll}. It is important to add this extra restriction on wholly negative pluralities, otherwise we risk trivializing statements such as "zero boys lifted the piano".

(58) $[\![P_{coll}]\!] = \{X \in D^{\pm} \mid X^+ \in [P_{coll}]\} \cup \{X \in D^{\pm} \mid X^+ = \emptyset, \neg\exists X' \subseteq X^-, X' \in [P_{coll}]\}$

Again, assume the domain is $\{a, b, c\}$, and that this time $\{a\}, \{a, b\} \in$ [lifted the piano], then the polarized denotation for 'lifted the piano' corresponds to (59). Assuming that a, b, c are *the people*, this predicts that in this scenario it will be true to say, e.g., "exactly one person lifted the piano", "exactly two people lifted the piano", etc., and false to say, e.g., "zero people lifted the piano". Assuming that c is the only smart person, it will however be true to say that "zero smart people lifted the piano".

(59) $$[\![\text{lifted the piano}]\!] = \left\{ \begin{array}{c} a^+, c^-, \\ a^+ \oplus b^+, a^+ \oplus b^-, a^+ \oplus c^-, b^- \oplus c^-, \\ a^+ \oplus b^+ \oplus c^-, a^+ \oplus b^- \oplus c^- \end{array} \right\}$$

There is still an open problem: Buccola and Spector, 2016 claim that an existential entailment with modified numerals does emerge with collective predicates. To illustrate this point, Buccola and Spector (2016, p. 161) claim that (60) is false in the provided context:

(60) Context: *A group of boys want to see how many of them it takes to lift the piano. Different groups take turns trying to lift the piano, but in the end, no group manages to.* Less than four boys lifted the piano.

It's quite unclear how to account for this (Buccola and Spector 2016 themselves do not fully resolve this issue). It's not feasible to remove wholly negative pluralities from the predicate denotation, as collective predicates can compose with *zero* (cf. Bylinina and Nouwen 2018).

(61) Zero soldiers surround the castle.

However, it is not totally clear whether the existential inference here is really a semantic entailment. There are certain embedded environments in which it disappears (I'm grateful to Filipe Hisao Kobayashi for pointing this out). Consider (62)—in the special case where zero soldiers surround the castle, then (62) certainly conveys that Able will try to escape.

(62) If less than ten soldiers surround the castle, then Able will try to escape.

6.2. Semantic singularity

In the polarized fragment, plural quantificational expressions such as 'some boys' and 'all boys' are taken as the basic case. Some interesting issues arise when one considers how to extend the polarized fragment in order to capture the singular-plural distinction, in order to account for contrasts as in (63).

(63) a. Some boys gathered.
b. *Some boy gathered

One potential move could be to insist that *singular* quantificational expressions are really just generalized quantifiers over ordinary (positive) individuals. There are a number of reasons to be dissatisfied with this position however. For example, as demonstrated in section 3.4, maximal mixed-polarity pluralities provide sufficient expressive power to characterize all conservative generalized quantifiers as predicates. It therefore might be considered desirable to dispense with GQ-theoretic determiners as lexical entries altogether, relying instead on the more restricted meaning-space afforded by mixed-polarity pluralities.

This however makes it extremely difficult to account for the distinction, e.g., between singular 'some boy', and plural 'some boys', or indeed between singular 'each boy' and plural 'all boys'. Exactly the same kinds of problems arise in GQ-theory however, so the polarized fragment is at least not worse than GQ-theory in this respect.

7. Conclusion

In this paper, I've suggested that many of the major issues associated with a plural semantics for numeral expressions can be addressed by enriching the domain of pluralities, to include both positive and negative individuals. Moreover, quantificational expressions generally can be characterized just as predicates of such pluralities, on the assumption that nominal restrictors have a maximality component. A general lesson here is modified numerals, and cumulative readings in particular unavoidably require some non-standard assumptions. Existing accounts (Brasoveanu, 2013; Haslinger and Schmitt, 2020) primarily pay this price by introducing non-standard compositional mechanisms to sidestep scopal interactions. Here, I've shown that a different strategy can be pursued: by incorporating a notion of positive vs. negative information into the domain of pluralities, cumulative readings can be accounted for via standard compositional mechanisms. This approach has the virtue of simultaneously addressing a constellation of problems associated with the proper treatment of numeral expressions. In section 3.4, I briefly float the possibility that system developed in this paper could serve as a fully-fledged alternative to GQ-theory and higher-order functions, the otherwise prevailing approach to quantification since Montague, 1973. While this possibility is tantalizing, a great deal of work is necessary before it can be determined whether or not this is feasible.

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